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ON THE SPACE OF GENERALIZED THETA-SERIES FOR CERTAIN QUADRATIC FORMS IN ANY NUMBER OF VARIABLES

Ketevan Shavgulidze

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ABSTRACT. An upper bound of the dimension of vector spaces of generalized theta-series corresponding to some nondiagonal quadratic forms in any number of variables is established. In a number of cases, an upper bound of the dimension of the space of theta-series with respect to the quadratic forms of five variables is improved and the basis of this space is constructed.

1. Introduction

Let

$$Q(X) = Q(x_1, x_2, \dots, x_r) = \sum_{1 \le i \le j \le r} b_{ij} x_i x_j$$

be an integer positive definite quadratic form of r variables and let $A = (a_{ij})$ be the symmetric $r \times r$ matrix of the quadratic form $Q(X)$, where $a_{ii} = 2b_{ii}$ and $a_{ij} = a_{ji} = b_{ij}$, for $i < j$. If $X = (x_1 \dots x_r)^T$ denotes a column matrix and X^T its transpose, then $Q(X) = \frac{1}{2} X^T A X$. Let A_{ij} denote the cofactor to the element a_{ij} in A and a^*_{ij} is the element of the inverse matrix A^{-1}.

A homogeneous polynomial $P(X) = P(x_1, \dots, x_r)$ of degree ν with complex coefficients, satisfying the condition

$$\sum_{1 \le i,j \le r} a^*_{ij} \Big(\frac{\partial^2 P}{\partial x_i \partial x_j} \Big) = 0 \tag{1.1}$$

is called a spherical polynomial of order ν with respect to $Q(X)$ (see [4]).

Let $\mathcal{P}(\nu, Q)$ denote the vector space over $\mathbb{C}$ of spherical polynomials $P(X)$ of even order ν with respect to $Q(X)$.

Hecke [6] calculated the dimension of the space $\mathcal{P}(\nu, Q)$ and showed that

$$\dim \mathcal{P}(\nu, Q) = \binom{\nu + r - 1}{r - 1} - \binom{\nu + r - 3}{r - 1}.$$

He formed a basis of the space of spherical polynomials of second order ($\nu = 2$) with respect to $Q(X)$.

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Lomadze [7] constructed a basis of the space of spherical polynomials of fourth order ($\nu = 4$) with respect to $Q(X)$. In the next section a basis of the space $\mathcal{P}(\nu, Q)$ is constructed with a simpler way.

Let

$$\vartheta(\tau, P, Q) = \sum_{n \in \mathbb{Z}^r} P(n) z^{Q(n)}, \qquad z = \mathrm{e}^{2\pi \mathrm{i}\tau}, \qquad \tau \in \mathbb{C}, \qquad \operatorname{Im} \tau > 0$$

be the corresponding generalized r-fold theta-series. Schoeneberg [8] proved that function $\vartheta(\tau, P, Q)$ is a modular form of weight $-(\frac{r}{2} + \nu)$ with respect to congruence subgroup $\Gamma_0(N)$, where N is the least positive integer such that NA^{-1} is again the even integral symmetric matrix. The map which assigns to each P in $\mathcal{P}(\nu, Q)$ the modular form $\vartheta(\tau, P, Q)$ is a linear transformation.

Let $T(\nu, Q)$ denote the vector space over $\mathbb{C}$ of generalized multiple theta-series, i.e.,

$$T(\nu, Q) = \{\vartheta(\tau, P, Q) \; : \; P \in \mathcal{P}(\nu, Q)\}.$$

Gooding [4,5] calculated the dimension of the vector space $T(\nu, Q)$ for reduced binary quadratic form Q and obtained an upper bound of the dimension of the space $T(\nu, Q)$ for some diagonal quadratic form of r variables

$$\dim T(\nu, Q) \leq \binom{\frac{\nu}{2} + r - 2}{r - 2}. \tag{1.2}$$

In [9, 10], the upper bounds for the dimensions of the spaces $T(\nu, Q)$ for some ternary and quaternary quadratic forms are established, in a number of cases the dimensions are calculated and the bases of these spaces are formed.

Gaigalas [1]–[3] obtained the upper bounds for the dimensions of the spaces $T(4, Q)$ and $T(6, Q)$ for some diagonal quadratic forms and presented the upper bounds of the dimensions of the spaces $T(\nu, Q)$ for some diagonal quadratic forms of six variables.

In this paper the upper bounds for the dimensions of the spaces $T(\nu, Q)$ for some nondiagonal quadratic forms of any number of variables are obtained. The dimension of the space $T(2, Q)$ is calculated and a basis of this space is constructed. In a number of cases the upper bounds for the dimensions of the spaces $T(\nu, Q)$ for some quadratic forms of five variables are improved.

In the sequel we use the following definition and results:

An integral $r \times r$ matrix U called an integral automorphism of the quadratic form $Q(X)$ in r variables if $U^T A U = A$.

Lemma 1.1 ([4: p. 37]). *Let $Q(X) = Q(x_1, \dots, x_r)$ be a positive definite quadratic form in r variables and $P(X) = P(x_1, \dots, x_r) \in P(\nu, Q)$. Let G be the set of all integral automorphisms of Q. Suppose*

$$\sum_{i=1}^{t} P(U_i X) = 0 \qquad \textit{for some} \;\; U_1, \dots, U_t \subseteq G,$$

then $\vartheta(\tau, P, Q) = 0$.

2. The basis of the space $\mathcal{P}(\nu, Q)$

Let

$$P(X) = P(x_1, x_2, x_3, \dots, x_r) = \sum_{k=0}^{\nu} \sum_{i=0}^{k} \sum_{j=0}^{i} \dots \sum_{l=0}^{m} a_{kij\dots l} x_1^{\nu-k} x_2^{k-i} x_3^{i-j} \dots x_r^{l}$$

be a spherical function of order ν with respect to the positive quadratic form $Q(x_1, x_2, x_3, \dots, x_r)$ of r variables and

$$L = (a_{000\dots0}, a_{100\dots0}, a_{110\dots0}, a_{111\dots0}, \dots, a_{\nu\nu\nu\dots\nu})^T$$

be the column vector, where $a_{kij\dots l}$ $(\nu \geq k \geq i \geq j \geq \dots \geq l \geq 0)$ are the coefficients of polynomial $P(X)$.

According to (1.1), the condition

$$\frac{1}{|A|} \sum_{1\leq i,j\leq r} A_{ij}\Big(\frac{\partial^2 P}{\partial x_i \partial x_j}\Big) = 0$$

is satisfied. Considering

$$\begin{aligned}\frac{\partial^2 P}{\partial x_1^2} &= \sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\dots\sum_{l=0}^{m}(\nu-k)(\nu-k-1)a_{kij\dots l}x_1^{\nu-k-2}x_2^{k-i}x_3^{i-j}\dots x_r^{l} \\ &= \sum_{k=1}^{\nu-1}\sum_{i=0}^{k-1}\sum_{j=0}^{i}\dots\sum_{l=0}^{m}(\nu-k+1)(\nu-k)a_{k-1ij\dots l}x_1^{\nu-k-1}x_2^{k-i-1}x_3^{i-j}\dots x_r^{l}\end{aligned}$$

and also obtain similar formulas for other second partial derivatives, then condition (1.1) takes the form

$$\begin{aligned}\frac{1}{|A|}\sum_{k=1}^{\nu-1}\sum_{i=0}^{k-1}\sum_{j=0}^{i}\dots\sum_{l=0}^{m}\big(&A_{11}(\nu-k+1)(\nu-k)a_{k-1ij\dots l}+2A_{12}(\nu-k)(k-i)a_{kij\dots l} \\ &+2A_{13}(\nu-k)(i-j+1)a_{ki+1j\dots l}+2A_{14}(\nu-k)(j+1)a_{ki+1j+1\dots l} \\ &+\dots+A_{rr}(l+2)(l+1)a_{k+1i+2j+2\dots l+2}\big)x_1^{\nu-k-1}x_2^{k-i-1}x_3^{i-j}\dots x_r^{l}=0.\end{aligned}$$

It follows that condition (1.1) in matrix notation has the following form

$$S \cdot L = 0,$$

where the matrix S has the form

$$S = \begin{pmatrix} A_{11}\nu(\nu-1) & 2A_{12}(\nu-1) & 2A_{13}(\nu-1) & 2A_{14}(\nu-1) & \dots & \dots & \dots & 0 \\ 0 & A_{11}(\nu-1)(\nu-2) & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & A_{11}(\nu-1)(\nu-2) & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & A_{11}(\nu-1)(\nu-2) & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2A_{11} & \dots & A_{rr}(\nu-1)\nu \end{pmatrix}$$

and is $\binom{\nu+r-3}{r-1} \times \binom{\nu+r-1}{r-1}$ matrix (the number of rows of the matrix S is equal to the number of $(k,i,j,\dots,l)$ with $0 \leq l \leq \dots \leq j \leq i < k \leq \nu-1$, the number of columns is equal to the number of coefficients $a_{kij\dots l}$, i.e., to the number of $(k,i,j,\dots,l)$ with $0 \leq l \leq \dots \leq j \leq i \leq k \leq \nu$).

We partition the matrix S into two matrices S_1 and S_2, where S_1 is the left square nondegenerate $\binom{\nu+r-3}{r-1} \times \binom{\nu+r-3}{r-1}$ matrix, it consists of the first $\binom{\nu+r-3}{r-1}$ columns of the matrix S; the matrix S_2 consists of the last $\binom{\nu+r-1}{r-1} - \binom{\nu+r-3}{r-1}$ columns of the matrix S.

Similarly, we partition the matrix L into two matrices L_1 and L_2, where L_1 is the $\binom{\nu+r-3}{r-1} \times 1$ matrix, it consists of the upper $\binom{\nu+r-3}{r-1}$ elements of L; the matrix L_2 consists of the lower $\binom{\nu+r-1}{r-1} - \binom{\nu+r-3}{r-1}$ elements of the matrix L.

According to the new notation, the matrix equality has the form

$$S_1L_1 + S_2L_2 = 0, \qquad \text{i.e.,} \qquad L_1 = -S_1^{-1}S_2L_2.$$

It follows from this equality that the matrix L_1 is expressed through the matrix L_2, and consequently, the first $\binom{\nu+r-3}{r-1}$ elements of the matrix L are expressed through its other elements.

Since the matrix L consists of the coefficients of the spherical polynomial $P(X)$, its first $\binom{\nu+r-3}{r-1}$ coefficients can be expressed through the last $\binom{\nu+r-1}{r-1}-\binom{\nu+r-3}{r-1}$ coefficients.

Hence, the polynomials

$$\begin{aligned} &P_{\nu-1,00\dots0}(a^{(1)}_{000\dots0}, a^{(1)}_{100\dots0}, \dots, a^{(1)}_{\nu-2,\nu-2,\nu-2,\dots,\nu-2}, 1, 0, 0, \dots, 0),\\ &P_{\nu-1,10\dots0}(a^{(2)}_{000\dots0}, a^{(2)}_{100\dots0}, \dots, a^{(2)}_{\nu-2,\nu-2,\nu-2,\nu-2}, 0, 1, 0, \dots, 0),\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &P_{\nu,\nu,\nu\dots\nu}(a^{(t)}_{000\dots0}, a^{(t)}_{100\dots0}, \dots, a^{(t)}_{\nu-2,\nu-2,\nu-2,\dots,\nu-2}, 0, 0, 0, \dots, 1), \end{aligned} \tag{2.1}$$

where the first $\binom{\nu+r-3}{r-1}$ coefficients from $a_{000\dots0}$ to $a_{\nu-2,\nu-2,\nu-2,\dots,\nu-2}$ are calculated through other $\binom{\nu+r-1}{r-1}-\binom{\nu+r-3}{r-1}$ coefficients, form the basis of the space $\mathcal{P}(\nu,Q)$ (the coefficients of polynomial $P_{bc\dots d}$ are given in the brackets, $a_{bc\dots d}$ is equal to 1 and the rest of those coefficients for which b is $\nu-1$ or ν are equal to 0).

The construction of the matrix S and the spherical polynomials for small ν ($\nu=4$) and small r ($r=3$) are considered in Appendix.

3. On the dimension of $T(\nu,Q)$ for certain quadratic forms in any number of variables

Consider the generalized r-fold theta-series

$$\vartheta(\tau,P,Q)=\sum_{n\in\mathbb{Z}^r}P(n)z^{Q(n)}, \qquad z=\mathrm{e}^{2\pi i\tau}.$$

Our goal is to construct a basis of the space of generalized theta-series with spherical polynomial P of order ν for quadratic form of r variables.

Let

$$Q(X)=b_{11}x_1^2+b_{22}x_2^2+b_{33}x_3^2+\dots+b_{rr}x_r^2$$

($0<b_{11}<b_{22}<\dots<b_{rr}$) be the diagonal quadratic form of r variables. The integral automorphisms of the quadratic form $Q(X)$ are

$$U=\begin{pmatrix} e_1 & 0 & 0 & \dots & 0\\ 0 & e_2 & 0 & \dots & 0\\ 0 & 0 & e_3 & \dots & 0\\ . & . & . & \dots & .\\ 0 & 0 & 0 & \dots & e_r \end{pmatrix}, \tag{3.1}$$

where $e_i=\pm1$.

We now consider all possible polynomials $P=P_{bc\dots d}(U_jX)$, where P is a spherical polynomial of order ν with respect to $Q(X)$, $P\in\mathcal{P}(\nu,Q)$ and U_j is an integral automorphism of the quadratic form $Q(X)$. We have to find which polynomials P satisfy equality

$$\sum_{i=1}^{t}P(U_iX)=0 \qquad \text{for some } U_1,\dots,U_t\in G.$$

For such polynomials according to Lemma 1.1, $\vartheta(\tau,P,Q)=0$.

For example, if

$$U_1 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & -1 \end{pmatrix},$$

then

$$P_{kij\dots l}(X) + P_{kij\dots l}(U_1 X) = \sum_{k=0}^{\nu} \sum_{i=0}^{k} \sum_{j=0}^{i} \dots \sum_{l=0}^{m} (1 + (-1)^l) a_{kij\dots l} x_1^{\nu-k} x_2^{k-i} x_3^{i-j} \dots x_r^l.$$

Equality

$$P_{kij\dots l}(X) + P_{kij\dots l}(U_1 X) = 0$$

takes place if and only if the condition

$$\left(1 + (-1)^l\right) a_{kij\dots l} = 0$$

is satisfied, that means the index l of the coefficient equal to one is odd. Similarly, it follows that if among the last $\binom{\nu+r-1}{r-1} - \binom{\nu+r-3}{r-1}$ coefficients of P, at least one of indices $k, i, j, \dots, l$ of the coefficient, equaled to one, is odd, then the spherical polynomial $P = P_{kij\dots l}$ satisfies the equality $\vartheta(\tau, P, Q) = 0$. Hence, if theta-series are linearly independent, then the indices $k, i, j, \dots, l$ of the corresponding spherical polynomial P are even. Thus the maximal number of linearly independent theta-series is

$$\sum_{i=0,\,2|i}^{\nu} \sum_{j=0,\,2|j}^{i} \dots \sum_{m=0,\,2|m}^{s} \sum_{l=0,\,2|l}^{m} 1 = \sum_{i=0,\,2|i}^{\nu} \sum_{j=0,\,2|j}^{i} \dots \sum_{m=0,\,2|m}^{s} \left(\frac{m}{2} + 1\right) = \binom{\frac{\nu}{2} + r - 2}{r - 2};$$

here $k = \nu$ is even.

We have shown inequality (1.2) and also showed that a basis of the space $T(\nu, Q)$ is among the theta-series $\vartheta(\tau, P, Q)$ with spherical polynomial $P = P_{kij\dots l}$ with even indices $k = \nu, i, j, \dots l$.

Consider now the nondiagonal quadratic form

$$Q_1(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + \dots + b_{rr}x_r^2 + b_{12}x_1x_2,$$

where $0 < |b_{12}| < b_{11} < b_{22} < \dots < b_{rr}$. The integral automorphisms of the quadratic form $Q_1(X)$ are

$$\begin{pmatrix} e_1 & 0 & 0 & \dots & 0 \\ 0 & e_1 & 0 & \dots & 0 \\ 0 & 0 & e_2 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & e_{r-1} \end{pmatrix}, \tag{3.2}$$

where $e_i = \pm 1$. Consider all possible polynomials $P_{kij\dots l}(U_h X)$, where $P_{kij\dots l}$ are spherical polynomials of order ν with respect to $Q_1(X)$, $P_{kij\dots l} \in \mathcal{P}(\nu, Q_1)$ and U_h is an integral automorphism of the quadratic form $Q_1(X)$. We have to find which polynomials $P_{kij\dots l}$ satisfy equality

$$P_{kij\dots l}(X) + P_{kij\dots l}(U_h X) = 0. \tag{3.3}$$

Equality (3.3) takes place if and only if the coefficients satisfy the equality

$$(1 + (-1)^w)\, a_{kij\dots l} = 0, \tag{3.4}$$

where $w = i, j, \dots, l$. Using the construction of a basis of the space of spherical functions, it is sufficient to show that (3.4) is true for the last coefficients from $a_{\nu-1,0,0\dots0}$ to $a_{\nu\nu\nu\dots\nu}$, i.e., when $k = \nu - 1, \nu$; $i = 0, 1, \dots, k$, $j = 0, 1, \dots, i, \dots,$ $l = 0, 1, \dots, m$. It follows that if among the last coefficients $a_{kij\dots l}$ of P, for which k is $\nu-1$ or ν at least one of indices $i, j, \dots, l$ is odd, then for the spherical polynomial $P = P_{kij\dots l}$ we get $\vartheta(\tau, P, Q) = 0$.

Hence, if theta-series are linearly independent, then the indices $i, j, \dots, l$ of the corresponding spherical polynomial P is even. Calculate how many coefficients $a_{kij\dots l}$ with $\nu-1 \le k \le \nu, 0 \le i \le k, \dots, 0 \le l \le m$ have the indices $i, j \dots$ or l even. We have the following cases:

a) for $k = \nu - 1$ we obtain

$$\sum_{i=0,\,2|i}^{\nu-2}\ \sum_{j=0,\,2|j}^{i}\cdots\sum_{m=0,\,2|m}^{s}\ \sum_{l=0,\,2|l}^{m} 1 = \sum_{i=0,\,2|i}^{\nu-2}\ \sum_{j=0,\,2|j}^{i}\cdots\sum_{m=0,\,2|m}^{s}\left(\frac{m}{2}+1\right) = \binom{\frac{\nu-2}{2}+r-2}{r-2}.$$

b) for $k = \nu$ it follows that

$$\sum_{i=0,\,2|i}^{\nu}\ \sum_{j=0,\,2|j}^{i}\cdots\sum_{m=0,\,2|m}^{s}\ \sum_{l=0,\,2|l}^{m} 1 = \binom{\frac{\nu}{2}+r-2}{r-2}.$$

Hence, the maximal number of theta-series with corresponding spherical polynomial $P_{kij\dots l}(X)$ with even indices $i, j, \dots, l$ is

$$\binom{\frac{\nu-2}{2}+r-2}{r-2} + \binom{\frac{\nu}{2}+r-2}{r-2} = \frac{1}{r-2}\binom{\frac{\nu}{2}+r-3}{r-3}(\nu+r-2).$$

Thus we have the following

Theorem 3.1. *Let $Q_1(X)$ be the nondiagonal quadratic form of r variables, given by $Q_1(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + \dots + b_{rr}x_r^2 + b_{12}x_1x_2$ then*

$$\dim T(\nu, Q_1) \le \frac{1}{r-2}\binom{\frac{\nu}{2}+r-3}{r-3}(\nu+r-2). \tag{3.5}$$

We now construct the basis of the space $T(\nu, Q_1)$, when $\nu = 2$. For quadratic form $Q_1(X)$ we have

$$|A| = \det A = 2^{r-2}(4b_{11}b_{22} - b_{12}^2)b_{33}\dots b_{rr}, \qquad a_{11}^* = \frac{2b_{22}}{4b_{11}b_{22} - b_{12}^2},$$

$$a_{12}^* = a_{21}^* = -\frac{b_{12}}{4b_{11}b_{22} - b_{12}^2}, \qquad a_{22}^* = \frac{2b_{11}}{4b_{11}b_{22} - b_{12}^2}, \quad a_{33}^* = \frac{1}{2b_{33}},$$

$$a_{44}^* = \frac{1}{2b_{44}}, \dots, \quad a_{rr}^* = \frac{1}{2b_{rr}}, \quad \text{and other} \quad a_{ij}^* = 0 \ \text{for} \ \ i \ne j.$$

It is easy to verify, that the spherical polynomials (2.1) of second order:

$$P_{100\dots0} = \frac{b_{12}}{2b_{22}}x_1^2 + x_1x_2, \qquad P_{110\dots0} = x_1x_3,$$

$$P_{1110\dots0} = x_1x_4, \qquad \dots, \qquad P_{200\dots0} = -\frac{b_{11}}{b_{22}}x_1^2 + x_2^2,$$

$$P_{210\dots0} = x_2x_3, \qquad P_{220\dots0} = -\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{33}}x_1^2 + x_3^2, \qquad P_{221\dots0} = x_3x_4,$$

$$P_{222\dots0} = -\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{44}}x_1^2 + x_4^2, \qquad \dots, \qquad P_{222\dots2} = -\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{rr}}x_1^2 + x_r^2$$

form the basis of the space of spherical polynomials of second order with respect to $Q_1(x)$.

Now we construct the corresponding generalized theta-series. Consider all possible polynomials $P_{kij\dots l}$, with even indices $i, j, \dots, l$ and $k = \nu - 1, \nu$; their number is r:

$$\vartheta(\tau, P_{100\dots0}, Q_1) = \sum_{n=1}^{\infty}\Big(\sum_{Q_1(x)=n} P_{100\dots0}(x)\Big)z^n = \sum_{n=1}^{\infty}\Big(\sum_{Q_1(x)=n}\big(\frac{b_{12}}{2b_{22}}x_1^2 + x_1x_2\big)\Big)z^n$$
$$= \frac{b_{12}}{b_{22}}z^{b_{11}} + \dots + 0z^{b_{22}} + \dots + 0z^{b_{33}} + \dots + 0z^{b_{44}} + \dots + 0z^{b_{rr}} + \dots,$$
$$\vartheta(\tau, P_{200\dots0}, Q_1) = \sum_{n=1}^{\infty}\Big(\sum_{Q_1(x)=n} P_{200\dots0}(x)\Big)z^n = \sum_{n=1}^{\infty}\Big(\sum_{Q_1(x)=n}\big(-\frac{b_{11}}{b_{22}}x_1^2 + x_2^2\big)\Big)z^n$$
$$= -\frac{2b_{11}}{b_{22}}z^{b_{11}} + \dots + 2z^{b_{22}} + \dots + 0z^{b_{33}} + \dots + 0z^{b_{44}} + \dots + 0z^{b_{rr}} + \dots,$$
$$\vartheta(\tau, P_{220\dots0}, Q_1) = \sum_{n=1}^{\infty}\Big(\sum_{Q_1(x)=n} P_{220\dots0}(x)\Big)z^n = \sum_{n=1}^{\infty}\Big(\sum_{Q_1(x)=n}\big(-\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{33}}x_1^2 + x_3^2\big)\Big)z^n$$
$$= -\frac{4b_{11}b_{22} - b_{12}^2}{2b_{22}b_{33}}z^{b_{11}} + \dots + 0z^{b_{22}} + \dots + 2z^{b_{33}} + \dots + 0z^{b_{44}} + \dots + 0z^{b_{rr}} + \dots,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\vartheta(\tau, P_{222\dots2}, Q_1) = \sum_{n=1}^{\infty}\Big(\sum_{Q_1(x)=n} P_{222\dots2}(x)\Big)z^n = \sum_{n=1}^{\infty}\Big(\sum_{Q_1(x)=n}\big(-\frac{4b_{11}b_{22} - b_{12}^2}{4b_{22}b_{rr}}x_1^2 + x_r^2\big)\Big)z^n$$
$$= -\frac{4b_{11}b_{22} - b_{12}^2}{2b_{22}b_{rr}}z^{b_{11}} + \dots + 0z^{b_{22}} + \dots + 0z^{b_{33}} + \dots + 0z^{b_{44}} + \dots + 2z^{b_{rr}} + \dots.$$

These generalized theta-series are linearly independent since the determinant constructed from the coefficients of these theta-series is not equal to zero. By virtue of (3.5) we have $\dim T(2, Q_1) \le r$. Hence these theta-series form the basis of the space $T(2, Q_1)$. We have the following

Theorem 3.2. *Let $Q_1(X)$ be the nondiagonal quadratic form of r variables, given by $Q_1(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + \dots + b_{rr}x_r^2 + b_{12}x_1x_2$, then $\dim T(2, Q_1) = r$ and the generalized theta-series with spherical polynomials $P_{kij\dots l}$ ($k = 1$ or 2; $i, j, \dots, l$ are even):*

$$\vartheta(\tau, P_{100\dots0}, Q_1),\quad \vartheta(\tau, P_{200\dots0}, Q_1),\quad \vartheta(\tau, P_{220\dots0}, Q_1),\quad \vartheta(\tau, P_{222\dots0}, Q_1),\ \dots,\ \vartheta(\tau, P_{222\dots2}, Q_1)$$

form the basis of the space $T(2, Q_1)$.

Now consider the quadratic form of five variables

$$Q_2 = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}(x_4^2 + x_5^2),$$

where $0 < b_{11} < b_{22} < b_{33} < b_{44} = b_{55}$.

We construct the integral automorphisms U of the quadratic form Q_2. Since

$$b_{11} = Q_2(\pm1, 0, 0, 0, 0), \qquad b_{22} = Q_2(0, \pm1, 0, 0, 0), \qquad b_{33} = Q_2(0, 0, \pm1, 0, 0),$$
$$b_{44} = b_{55} = Q_2(0, 0, 0, \pm1, 0) = Q_2(0, 0, 0, 0, \pm1).$$

It is easy to verify that the integral automorphisms of the quadratic form Q_2 are the automorphisms (3.1) and the automorphisms

$$\begin{pmatrix} e_1 & 0 & 0 & 0 & 0 \\ 0 & e_2 & 0 & 0 & 0 \\ 0 & 0 & e_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_4 \\ 0 & 0 & 0 & e_5 & 0 \end{pmatrix} \qquad (e_i = \pm1, \quad i = 1, 2, 3, 4, 5). \tag{3.6}$$

From the automorphisms (3.6) we use only

$$U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.7}$$

The automorphisms of the quadratic form Q are also automorphisms of the quadratic form Q_2. Therefore using (1.2) we have

$$\dim T(\nu, Q_2) \le \binom{\frac{\nu}{2}+r-2}{r-2} = \binom{\frac{\nu}{2}+3}{3}. \tag{3.8}$$

We improve this estimation. From (3.7) we have

$$P(U_2X)=\sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j}a^{(h)}_{kij\,l}x_1^{\nu-k}x_2^{k-i}x_3^{i-j}x_5^{j-l}x_4^{l}=\sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j}a^{(h)}_{kij\,l}x_1^{\nu-k}x_2^{k-i}x_3^{i-j}x_4^{j-(j-l)}x_5^{j-l}.$$

From here it follows that if all the last coefficients of the basis polynomial $P(X)$ are equal to zero, except one $a^{(h)}_{kijl}=1$, then all the last coefficients of polynomial $P(U_2X)$ are equal to zero, except one $a^{(s)}_{k,i,j,j-l}=1$. Hence, $P_{kijl}(U_2X)=P_{kijj-l}(X)$ is a basis polynomial of the space $\mathcal{P}(\nu,Q_2)$. Further, it is known ([4: p. 38]) that

$$\vartheta(\tau,P(X),Q_2)=\vartheta(\tau,P(U_2X),Q_2),$$

thus the theta-series $\vartheta(\tau,P(X),Q_2)$ and $\vartheta(\tau,P(U_2X),Q_2)$, corresponding to different basis polynomials $P(X)=P_{kijl}(X)$ and $P(U_2X)=P_{kijj-l}(X)$, are linearly dependent.

Calculate how many such linearly dependent theta-series we have. Let k,i,j and l be even (otherwise, it can be shown similarly to $Q(x)$ that $\vartheta(\tau,P,Q_2)=0$), i.e., $k=\nu$, $2|i$, $2|j$, $2|l$ and l takes

$$\sum_{2|l,\,l=0}^{j} 1 = \frac{j}{2}+1$$

even values for each even j. Hence we have

$$\left[\frac{1}{2}\Big(\frac{j}{2}+1\Big)\right] = \begin{cases} \dfrac{j}{4} & \text{if } j\equiv 0\,(\mathrm{mod}4), \\[2mm] \dfrac{j+2}{4} & \text{if } j\equiv 2\,(\mathrm{mod}4) \end{cases}$$

linearly dependent theta-series for each even j. Similarly, for each even i we have

$$\sum_{j=0,\,j\equiv 0(\mathrm{mod}4)}^{i}\frac{j}{4}+\sum_{j=0,\,j\equiv 2(\mathrm{mod}4)}^{i}\frac{j+2}{4} = \begin{cases} (1+\frac{i}{4})\frac{i}{4} & \text{if } i\equiv 0(\mathrm{mod}4), \\[2mm] \Big(\dfrac{i+2}{4}\Big)^2 & \text{if } i\equiv 2(\mathrm{mod}4) \end{cases}$$

linearly dependent theta-series. The number of linearly dependent theta-series for even ν is

$$\sum_{i=0,\,i\equiv 0(\mathrm{mod}4)}^{\nu}\Big(1+\frac{i}{4}\Big)\frac{i}{4}+\sum_{i=0,\,i\equiv 2(\mathrm{mod}4)}^{\nu}\Big(\frac{i+2}{4}\Big)^2 = \begin{cases} \dfrac{1}{24}\Big(\dfrac{\nu}{4}+1\Big)\nu(\nu+5) & \text{if } \nu\equiv 0(\mathrm{mod}4), \\[2mm] \dfrac{1}{24}\Big(\dfrac{\nu}{2}+1\Big)\Big(\dfrac{\nu}{2}+3\Big)(\nu+1) & \text{if } \nu\equiv 2(\mathrm{mod}4). \end{cases} \tag{3.9}$$

Hence, from (3.8) for the maximal number of linearly independent theta-series we get

$$\dim T(\nu, Q_2) \leq \begin{cases} \binom{\frac{\nu}{2}+3}{3} - \frac{1}{24}\left(\frac{\nu}{4}+1\right)\nu(\nu+5) & \text{if } \nu \equiv 0 (\text{mod} 4), \\ \binom{\frac{\nu}{2}+3}{3} - \frac{1}{24}\left(\frac{\nu}{2}+1\right)\left(\frac{\nu}{2}+3\right)(\nu+1) & \text{if } \nu \equiv 2 (\text{mod} 4). \end{cases}$$

Thus, we have the following theorem.

Theorem 3.3. *Let $Q_2(X)$ be the diagonal quadratic form of five variables, given by $Q_2(X) = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}(x_4^2 + x_5^2)$, then*

$$\dim T(\nu, Q_2) \leq \begin{cases} \frac{1}{6}\left(\frac{\nu}{4}+1\right)\left(\frac{\nu}{4}+2\right)(\nu+3) & \text{if } \nu \equiv 0 (\text{mod} 4), \\ \frac{1}{24}\left(\frac{\nu}{2}+1\right)\left(\frac{\nu}{2}+3\right)(\nu+7) & \text{if } \nu \equiv 2 (\text{mod} 4). \end{cases}$$

Similarly, consider the nondiagonal quadratic form of five variables

$$Q_3 = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + b_{44}(x_4^2 + x_5^2) + b_{12}x_1x_2,$$

where $0 < |b_{12}| < b_{11} < b_{22} < b_{33} < b_{44} = b_{55}$.

We construct the integral automorphisms U of the quadratic form Q_3. It is easy to verify that the integral automorphisms of the quadratic form Q_3 are the automorphisms (3.2) and the following automorphisms

$$\begin{pmatrix} e_1 & 0 & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 & 0 \\ 0 & 0 & e_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_3 \\ 0 & 0 & 0 & e_4 & 0 \end{pmatrix} \quad (e_i = \pm 1, \quad i = 1, 2, 3, 4). \tag{3.10}$$

From automorphisms (3.10) we use only U_2 (see (3.7)).

The automorphism of the quadratic form Q_1 are also the automorphisms of the quadratic form Q_3, therefore using (3.5) we have

$$\dim T(\nu, Q_3) \leq \frac{1}{r-2}\binom{\frac{\nu}{2}+r-3}{r-3}(\nu+r-2) = \frac{\nu+3}{3}\binom{\frac{\nu}{2}+2}{2}. \tag{3.11}$$

We improve this estimation. From (3.7) it follows that

$$P(U_2X) = \sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j} a_{kij\,l}^{(h)} x_1^{\nu-k}x_2^{k-i}x_3^{i-j}x_5^{j-l}x_4^{l} = \sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j} a_{kij\,l}^{(h)} x_1^{\nu-k}x_2^{k-i}x_3^{i-j}x_4^{j-(j-l)}x_5^{j-l}.$$

From here, $P(U_2X)$ is a basis polynomial of the space $\mathcal{P}(\nu, Q_3)$. Further, it is known ([4: p. 38]) that

$$\vartheta(\tau, P(X), Q_3) = \vartheta(\tau, P(U_2X), Q_3).$$

Thus the theta-series $\vartheta(\tau, P(X), Q_3)$ and $\vartheta(\tau, P(U_2X), Q_3)$, corresponding to the different basis polynomials $P(X) = P_{kijl}(X)$ and $P(U_2X) = P_{kijj-l}(X)$, are linearly dependent.

Calculate how many such linearly dependent theta-series we have. Let i, j and l be even (otherwise, it can be shown similarly to $Q(x)$ that $\vartheta(\tau, P, Q_3) = 0$), i.e., $2|i,\ 2|j,\ 2|l$.

For $k = \nu - 1$ the number of linearly dependent theta-series is

$$\sum_{i=0,\, i\equiv 0 (\text{mod} 4)}^{\nu-2}\left(1+\frac{i}{4}\right)\frac{i}{4} + \sum_{i=0,\, i\equiv 2 (\text{mod} 4)}^{\nu-2}\left(\frac{i+2}{4}\right)^2 = \begin{cases} \frac{1}{24}\frac{\nu}{2}\left(\frac{\nu}{2}+2\right)(\nu-1) & \text{if } \nu \equiv 0 (\text{mod} 4), \\ \frac{1}{24}\left(\frac{\nu}{2}-1\right)\left(\frac{\nu}{2}+1\right)(\nu+3) & \text{if } \nu \equiv 2 (\text{mod} 4). \end{cases}$$

For $k=\nu$ for the number of linearly dependent theta-series we have estimation (3.9).

Hence the number of linearly dependent theta-series for $k=\nu-1$ and $k=\nu$ all is

$$\frac{\nu}{12}\Big(\frac{\nu}{2}+1\Big)\Big(\frac{\nu}{2}+2\Big)$$

and for the maximal number of linearly independent theta-series from (3.11) we get

$$\dim T(\nu,Q_3)\leq\frac{\nu+3}{3}\binom{\frac{\nu}{2}+2}{2}-\frac{\nu}{12}\Big(\frac{\nu}{2}+1\Big)\Big(\frac{\nu}{2}+2\Big)=\binom{\frac{\nu}{2}+3}{3}.$$

Thus we have the following

Theorem 3.4. *Let $Q_3(X)$ be the nondiagonal quadratic form of five variables, given by $Q_3(X)=b_{11}x_1^2+b_{22}x_2^2+b_{33}x_3^2+b_{44}(x_4^2+x_5^2)+b_{12}x_1x_2$, then*

$$\dim T(\nu,Q_3)\leq\binom{\frac{\nu}{2}+3}{3}.$$

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Appendix

Now consider a full example with small ν ($\nu=4$) and small r ($r=3$) to clarify the whole picture.

For quadratic form $Q_1(x_1,x_2,x_3)=b_{11}x_1^2+b_{22}x_2^2+b_{33}x_3^2+b_{12}x_1x_2$ we have $|A|=\det A=2b_{33}(4b_{11}b_{22}-b_{12}^2)$, $A_{11}=4b_{22}b_{33}$, $A_{12}=-2b_{12}b_{33}$, $A_{22}=4b_{11}b_{33}$, $A_{13}=A_{23}=0$, $A_{33}=4b_{11}b_{22}-b_{12}^2$.

Let

$$P(X)=P(x_1,x_2,x_3)=\sum_{k=0}^{\nu}\sum_{i=0}^{k}\sum_{j=0}^{i}a_{ki}x_1^{\nu-k}x_2^{k-i}x_3^{i}$$

be a spherical function of order ν with respect to the ternary quadratic form $Q_1(x_1, x_2, x_3)$ and

$$L = (a_{00}\ a_{10}\ a_{11}\ a_{20}\ a_{21}\ a_{22}\ a_{30}\ \dots\ a_{\nu\nu})^T$$

be a column vector, where a_{ki} $(\nu \geq k \geq i \geq 0)$ are the coefficients of polynomial $P(x_1, x_2, x_3)$.

The condition (1) for the quadratic form $Q_1(x_1, x_2, x_3)$ takes the form

$$\frac{1}{|A|}\sum_{k=1}^{\nu-1}\sum_{i=0}^{k-1}\big(A_{11}(\nu-k+1)(\nu-k)a_{k-1i} + 2A_{12}(\nu-k)(k-i)a_{ki}$$
$$+2A_{13}(\nu-k)(i+1)a_{ki+1} + A_{22}(k-i)(k-i+1)a_{k+1i}$$
$$+2A_{23}(k-i)(i+1)a_{k+1i+1} + A_{33}(i+2)(i+1)a_{k+1i+2}\big)x_1^{\nu-k-1}x_2^{k-i-1}x_3^{i} = 0.$$

In the matrix equation

$$S \cdot L = 0,$$

for $\nu = 4$ the matrix S has the following form

$$S = \begin{pmatrix} 12A_{11} & 6A_{12} & 0 & 2A_{22} & 0 & 2A_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6A_{11} & 0 & 8A_{12} & 0 & 0 & 6A_{22} & 0 & 2A_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6A_{11} & 0 & 4A_{12} & 0 & 0 & 2A_{22} & 0 & 6A_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2A_{11} & 0 & 0 & 6A_{12} & 0 & 0 & 0 & 12A_{22} & 0 & 2A_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2A_{11} & 0 & 0 & 4A_{12} & 0 & 0 & 0 & 6A_{22} & 0 & 6A_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2A_{11} & 0 & 0 & 2A_{12} & 0 & 0 & 0 & 2A_{22} & 0 & 12A_{33} \end{pmatrix}.$$

Consider all possible polynomials P_{ki}, with even indices i and $k = \nu - 1, \nu$; their number is 5 for $\nu = 4$:

$$P_{30} = \frac{b_{12}(b_{12}^2 - 2b_{11}b_{22})}{4b_{22}^3}x_1^4 + \frac{b_{12}^2 - b_{11}b_{22}}{b_{22}^2}x_1^3x_2 + \frac{3b_{12}}{2b_{22}}x_1^2x_2^2 + x_1x_2^3,$$

$$P_{32} = \frac{b_{12}(b_{12}^2 - 4b_{11}b_{22})}{24b_{22}^2b_{33}}x_1^4 + \frac{b_{12}^2 - 4b_{11}b_{22}}{12b_{22}b_{33}}x_1^3x_2 + \frac{b_{12}}{2b_{22}}x_1^2x_3^2 + x_1x_2x_3^2,$$

$$P_{40} = -\frac{b_{11}(b_{12}^2 - b_{11}b_{22})}{b_{22}^3}x_1^4 - \frac{4b_{11}b_{12}}{b_{22}^2}x_1^3x_2 - \frac{6b_{11}}{b_{22}}x_1^2x_2^2 + x_2^4,$$

$$P_{42} = \frac{(b_{12}^2 - 4b_{11}b_{22})(b_{12}^2 - 2b_{11}b_{22})}{24b_{22}^3b_{33}}x_1^4 + \frac{b_{12}(b_{12}^2 - 4b_{11}b_{22})}{6b_{22}^2b_{33}}x_1^3x_2$$
$$+ \frac{b_{12}^2 - 4b_{11}b_{22}}{4b_{22}b_{33}}x_1^2x_2^2 - \frac{b_{11}}{b_{22}}x_1^2x_3^2 + x_2^2x_3^2,$$

$$P_{44} = \frac{(b_{12}^2 - 4b_{11}b_{22})^2}{16b_{22}^2b_{33}^2}x_1^4 + \frac{3(b_{12}^2 - 4b_{11}b_{22})}{2b_{22}b_{33}}x_1^2x_3^2 + x_3^4.$$

Now we construct the corresponding generalized theta-series:

$$l\vartheta(\tau, P_{30}, Q_1) = \sum_{n=1}^{\infty}\Big(\sum_{Q_1(x)=n} P_{30}(x)\Big)z^n = \frac{b_{12}(b_{12}^2 - 2b_{11}b_{22})}{2b_{22}^3}z^{b_{11}} + \dots + 0z^{b_{22}} + \dots$$
$$+ 0z^{b_{33}} + \dots + \frac{b_{12}(b_{12}^2 - 2b_{11}b_{22})}{b_{22}^3}z^{b_{11}+b_{33}} + \dots + 0z^{b_{22}+b_{33}} + \dots,$$

$$\vartheta(\tau,P_{32},Q_1)=\sum_{n=1}^{\infty}\Big(\sum_{Q_1(x)=n}P_{32}(x)\Big)z^n=\frac{b_{12}(b_{12}^2-4b_{11}b_{22})}{12b_{22}^2b_{33}}z^{b_{11}}+\dots+0z^{b_{22}}+\dots$$
$$+0z^{b_{33}}+\dots+\Big(\frac{b_{12}(b_{12}^2-4b_{11}b_{22})}{6b_{22}^2b_{33}}+\frac{2b_{12}}{b_{22}}\Big)z^{b_{11}+b_{33}}+\dots+0z^{b_{22}+b_{33}}+\dots,$$
$$\vartheta(\tau,P_{40},Q_1)=\sum_{n=1}^{\infty}\Big(\sum_{Q_1(x)=n}P_{40}(x)\Big)z^n=-\frac{2b_{11}(b_{12}^2-b_{11}b_{22})}{b_{22}^3}z^{b_{11}}+\dots+2z^{b_{22}}+\dots$$
$$+0z^{b_{33}}+\dots-\frac{4b_{11}(b_{12}^2-b_{11}b_{22})}{b_{22}^3}z^{b_{11}+b_{33}}+\dots+4z^{b_{22}+b_{33}}+\dots,$$
$$\vartheta(\tau,P_{42},Q_1)=\sum_{n=1}^{\infty}\Big(\sum_{Q_1(x)=n}P_{42}(x)\Big)z^n=\frac{(b_{12}^2-4b_{11}b_{22})(b_{12}^2-2b_{11}b_{22})}{12b_{22}^3b_{33}}z^{b_{11}}+\dots+0z^{b_{22}}+\dots$$
$$+0z^{b_{33}}+\dots+\Big(\frac{(b_{12}^2-4b_{11}b_{22})(b_{12}^2-2b_{11}b_{22})}{6b_{22}^3b_{33}}-\frac{4b_{11}}{b_{22}}\Big)z^{b_{11}+b_{33}}+\dots+4z^{b_{22}+b_{33}}+\dots,$$
$$\vartheta(\tau,P_{44},Q_1)=\sum_{n=1}^{\infty}\Big(\sum_{Q_1(x)=n}P_{44}(x)\Big)z^n=\frac{(b_{12}^2-4b_{11}b_{22})^2}{8b_{22}^2b_{33}^2}z^{b_{11}}+\dots+0z^{b_{22}}+\dots$$
$$+2z^{b_{33}}+\dots+4\Big(\frac{(b_{12}^2-4b_{11}b_{22})^2}{16b_{22}^2b_{33}^2}+\frac{3(b_{12}^2-4b_{11}b_{22})}{2b_{22}b_{33}}+1\Big)z^{b_{11}+b_{33}}+\dots$$
$$+4z^{b_{22}+b_{33}}+\dots.$$

These generalized theta-series are linearly independent since the determinant of the fifth order constructed from the coefficients of these theta-series is not equal to zero. By virtue of (8) we have $\dim T(4,Q_1)\le\frac{(r-1)(r+2)}{2}=5$. Hence these theta-series form the basis of the space $T(4,Q_1)$.

We have the following

Theorem A. *Let $Q_1(X)$ be the nondiagonal ternary quadratic form, given by $Q_1(X)=b_{11}x_1^2+b_{22}x_2^2+b_{33}x_3^2+b_{12}x_1x_2$, then $\dim T(4,Q_1)=5$ and the generalized theta-series with spherical polynomials P_{ki} ($k=3$ or 4; i is even):*

$$\vartheta(\tau,P_{30},Q_1);\vartheta(\tau,P_{32},Q_1);\vartheta(\tau,P_{40},Q_1);\vartheta(\tau,P_{42},Q_1);\vartheta(\tau,P_{44},Q_1)$$

form the basis of the space $T(4,Q_1)$.

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Faculty of Exact and Natural Sciences
I. Javakhishvili Tbilisi State University
University Str. 13
Tbilisi 0186
GEORGIA
E-mail: ketevan.shavgulidze@tsu.ge