The Clark–Haussmann–Ocone type formula for functionals of the Compensated Poisson Process

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In 1998, for normal martingales (the martingale M is said to be normal, if its predictable square characteristic is deterministic $\langle M,M\rangle_t = t$), Ma, Protter and Martin proved that: if $F \in D_{2,1}^M$, then the Clark-Haussmann-Ocone representation $F = EF + \int_{(0,T]} {}^p(D_t^M F)dM_t$ is valid, where $D_{2,1}^M$ denotes the space of square integrable functionals having the first order stochastic derivative, and ${}^p(D_t^M F)$ is the predictable projection of the stochastic derivative $D_t^M F$ of the functional F. In this case, unlike the Wiener's one, it is impossible to define in a generally adopted manner an operator of stochastic differentiation to obtain the structure of Sobolev space $D_{2,1}^M$. For example, one fails to define the space $D_{\alpha,1}^M$ $(1 \le \alpha < 2)$ in a commonly adopted manner (i.e., by closing a class of smooth functionals with respect to the corresponding norm). In [1] we defined the $D_{\alpha,1}^M$ $(1 < \alpha < 2)$ for the normal martingales and generalized the Clark-Haussmann-Ocone formula to functionals of this space.

Ma, Protter and Martin gave an example showing that two possible ways of determination of a stochastic derivative coincide if and only if the quadratic martingale characteristic $[M,M]$ is the deterministic function (as, for example, in the Wiener's case $[W,W]_t = t$). Consequently, the Clark-Haussmann-Ocone formula makes it impossible to construct explicitly the operator of the stochastic derivative of the functionals of the Compensated Poisson process (which, obviously, belongs to a class of normal martingales $\langle M,M\rangle_t = t$, but its quadratic variation is not deterministic, $[M,M]_t = M_t + t$), saying nothing on the construction of its predictable projection.

Let $(\Omega, \Im, (\Im_t)_{0\le t\le T}, P)$ be a filtered probability space satisfying the usual conditions. Assume that the standard Poisson process $N_t = M_t + t$ is given on it ($P_k := P(N_t = k) = e^{-t}t^k / k!,\ k = 0,1,\dots$) and that $\Im_t$ is generated by N ($\Im_t = \Im_t^N$), $\Im = \Im_T$. Let $Z^+ = \{0,1,2,\dots\}$, $\Delta_- f(k) = f(k) - f(k-1)$ ($f(k) = 0,\ k < 0$); $\Delta_-^n := (\Delta_-)^n$ and define Poisson-Charlier polynomials $\prod_n(k) = (-1)^n \Delta_-^n P_k / P_k$, $n \ge 1$; $\prod_0 = 1$ (It is known that the system of normalized Poisson-Charlier polynomials is a basis in $L_2(Z^+) := \{f : \sum_{k=0}^\infty f^2(k) < \infty\}$). Let $L_2^T := \{f : e^{-T}\sum_{k=0}^\infty f^2(k)T^k / k! < \infty\}$ (this is a Banach space with the basis $\{k^n e^{-T}T^k / k!\}$). Denote $\Delta_+ f(x) := f(x+1) - f(x)$ ($\Delta_+ f(M_T) := \Delta_+ f(x)|_{x=M_T}$).

Lemma. If $f(\cdot - T) \in L_2^T$ then the stochastic integral $\int_{(0,T]} E\{f(M_T) \mid \Im_{t-}\}dM_t$ is well defined.

Theorem. If $f \in L_2^T$ and for some positive $\varepsilon > 0$ number $f(\cdot - T) \in L_2^{(1+\varepsilon)T}$, then the stochastic integral below is well defined and the following representation is valid:

$$f(M_T) = Ef(M_T) + \int_{(0,T]} E\{\Delta_+ f(M_T - \Delta M_t) \mid \Im_t\}dM_t \qquad (P\text{-a.s.}).$$

References

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