ON THE LAZARD RING

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Abstract. We define a formal power series $\sum A_{ij}x^iy^j$ over the Lazard ring Λ and the formal group laws F_n, $n \geq 2$ over a quotient ring of Λ. For each F_n we construct a complex cobordism theory with singularities with the coefficient ring $\mathbb{Q}[p_1, \cdots, p_{2n}]$, with parameters p_i, $|p_i| = 2i$.

1. Introduction and statements

Let Λ be the universal Lazard ring and $F(x,y) = \sum a_{ij}x^iy^j$ be the universal formal group law defined over Λ [9]. By universal property of Λ, for any formal group law G over any commutative ring with unit R there is a unique ring homomorphism $r : \Lambda \to R$, such that $G(x,y) = \sum r(a_{ij})x^iy^j$.

The formal group law of geometric cobordism F_U was introduced in [12]. It is proved by D.Quillen [13] that the coefficient ring of complex cobordism $MU_*(pt) = \mathbb{Z}[x_1, x_2, ...]$, $|x_i| = 2i$ is naturally isomorphic as a graded ring to the universal Lazard ring. Following Quillen we will identify F_U with the universal Lazard formal group law.

Regardless of the geometric nature of geometric cobordism the computation of the complex cobordism ring $MU^*(X)$ for concrete spaces X is often a very difficult task. Various modifications of the theory of complex cobordism were introduced, for example cobordism theories with singularities [14, 15].

This note presents the formal group laws F_n, $n \geq 2$ over the quotient rings of the Lazard ring. By our Theorem 1.3 the formal group law F_n can be realized as cohomology theory with coefficient ring $\mathbb{Q}[p_1, \cdots, p_{2n}]$, for some parameters $|p_i| = 2i$.

The hope is that the cohomology theory, which implements F_n, does not lose too much information and at the same time is better calculated, because the coefficient ring is a graded field with finite parameters.

Let f and g be the exponent and the logarithm of F respectively [8]. Thus

$$F(x,y) = f(g(x) + g(y)).$$

Let $\omega(x)$ be the invariant differential form of F

$$\omega(x) := \frac{\partial F(x,y)}{\partial y}(x,0), \ \ \omega(x) = 1 + \sum_{i\geq 1} w_ix^i, \ \ w_i \in \Lambda. \tag{1.1}$$

The following series in $\Lambda[[x]]$ were defined in [1]

$$A(x,y) = \sum A_{ij}x^iy^j = F(x,y)(x\omega(y) - y\omega(x). \tag{1.2}$$

Then $F(x,y)$ can be rewritten in the following form

$$F(x,y) = \frac{A(x,y)}{x\omega(y) - y\omega(x)}. \tag{1.3}$$

2010 *Mathematics Subject Classification.* 33E05, 55N22.

Key words and phrases. Lazard ring, Formal group law.

The first author was supported by CNRS PICS N 7736, and by Shota Rustaveli NSF grant 217-614.

The second author was supported by CNRS PICS N 7736.

Theorem 1.1. *For any integer $i \geq 2$ there exit the series $W_i(x) \in \Lambda[[x]]$, such that the series $A(x,y)$ is expressed in the following form*

$$A(x,y) = \big(x\omega(y) + y\omega(x) - \omega'(0)xy\big)\big(x\omega(y) - y\omega(x)\big) + \sum_{i\geq 2}\big(\omega(x)W_i(x) - \omega(y)W_i(y)\big)(xy)^i.$$

The sum of initial three terms in Theorem 1.1 corresponds to the Abel formal group law of the form [11], [6]

$$F(x,y) = xR(y) + yR(x) + \alpha xy.$$

The universal Buchstaber formal group law [1],[2],[3],[5], [7] can be written as follows

$$F_B(x,y) = x\bar{\omega}(y) + y\bar{\omega}(x) - \bar{\omega}'(0)xy + \frac{\bar{\omega}(x)\bar{W}_2(x) - \bar{\omega}(y)\bar{W}_2(y)}{x\bar{\omega}(y) - y\bar{\omega}(x)}(xy)^2,$$

with interesting specializations [4] such as the Euler formal group law [10].

In Section 2 we prove the following statement which determines $A(x,y)$ modulo $(xy)^4$.

Proposition 1.2. *One has in $\Lambda[[x,y]]/(xy)^4$*

$$\begin{aligned} A(x,y) = &\big(x\omega(y) + y\omega(x) - \omega'(0)xy\big)\big(x\omega(y) - y\omega(x)\big) + \\ &\big(\omega(x)W_2(x) - \omega(y)W_2(y)\big)x^2y^2 + \\ &\big(\omega(x)W_3(x) - \omega(y)W_3(y)\big)x^3y^3, \end{aligned}$$

where

$$W_2(x) = \frac{\omega'(x) - \omega'(0)}{2x};$$

$$W_3(x) = \frac{\frac{1}{6}[\omega''(x)\omega(x) + \omega'^2(x) - \omega''(0) - \omega'^2(0)] - W_2(x)\omega(x) + \frac{1}{2}\omega''(0) + \frac{1}{12}x\omega'''(0)}{x^2}.$$

The Abel formal groups can be realized as cohomology theory. The formal group law F_B can be realized after localized out of 2 [11].

Let us consider the homomorphism from the Lazard ring to its quotient by the ideal generated by $A_{n+1\,j}$, $j > n+1$, for fixed $n \geq 2$. Let F_n be the formal group law classified by the quotient homomorphism

$$\phi_n : \Lambda \to \Lambda/(A_{n+1\,j}, j > n+1). \tag{1.4}$$

In this way we get the inverse system of the formal groups laws. The morphism from F_{n+1} to F_n is given by natural projection of the corresponding coefficient rings $\Lambda_{n+1} \to \Lambda_n$.

The formal group laws F_n can be realized but over the rationals. In particular $\Lambda \otimes \mathbb{Q} = \mathbb{Q}[\mathbf{CP}_1, \mathbf{CP}_2, \cdots]$. Theorem 1.3 says that for any $n \geq 2$ the projection (1.4) over the rationals

$$\rho_n = \phi_n \otimes \mathbb{Q} : \Lambda \otimes \mathbb{Q} \to \Lambda_n \otimes \mathbb{Q} \tag{1.5}$$

is the quotient map by the ideal generated by the elements

$$\{\mathbf{CP}_k - P_k\big(\mathbf{CP}_1, \cdots, \mathbf{CP}_{2n}\big),\ k > 2n\}, \tag{1.6}$$

where P_k are some polynomials. Moreover, (1.3) forms a regular sequence and therefore defines an exact cohomology theory $\mathcal{H}^*(n)$, the complex cobordism with singularities. The coefficient ring of $\mathcal{H}^*(n)(pt)$ equals to $\mathbb{Q}[p_1, p_2, \cdots, p_{2n}]$, where $p_i = \rho_n(\mathbf{CP}_i)$.

Theorem 1.3. *Let F_n be the formal group law over $\Lambda_n \otimes \mathbb{Q}$ in (1.5). Then F_n uniquely determined by the property that its classifying map ρ_n kills the coefficients of the series $W_{n+1}(x)$. Equivalently, for the invariant differential form of F_n*

$$\omega_{F_n}(x) = 1 + \sum v_i x^i,$$

the coefficient v_k, $k > 2n$ is decomposable in $v_1, \cdots, v_{2n}$.

2. Preliminaries and Proof of Theorem 1.2

Because of antisymmetry

$$\begin{aligned} A(x,y) =&x^2 - y^2 - A_{12}xy^2 + A_{12}x^2y \\ &- x^2(A_{23}y^3 + A_{24}y^4 + \cdots + A_{2i}y^i + \cdots) \\ &+ y^2(A_{23}x^3 + A_{24}x^4 + \cdots + A_{2i}x^i + \cdots) \\ &- x^3(A_{34}y^4 + A_{35}y^5 + \cdots + A_{3i}y^i + \cdots)+ \\ &+ y^3(A_{34}x^4 + A_{35}x^5 + \cdots + A_{3i}x^i + \cdots)\cdots \end{aligned}$$

where $A_{12} = w_1$, $A_{23} = \frac{1}{2}w_3$ in terms of the invariant differential form.

Let $\omega(x) = 1 + w_1x + w_2x^2 + \cdots$ as in (1.1), and f and g be the exponent and the logarithm of F.

Then we have by definition

$$f'(g(x)) = \omega(f(g(x))) = \omega(x), \qquad \text{as } f'(x) = 1/g'(f(x)) = \omega(f(x)), \tag{2.1}$$

$$f''(g(x) = \omega(x)\omega'(x), \qquad \text{as } f''(g(x)g'(x) = \omega'(x), \tag{2.2}$$

$$f'''(g(x)) = \omega^2(x)\omega(x)'' + \omega(x)\omega'^2(x), \qquad \text{as } f'''(g(x))g'(x) = (\omega'(x)\omega(x))', \tag{2.3}$$

$$g''(0) = -\omega'(0), \tag{2.4}$$

$$g'''(0) = -\omega(0)'' + 2\omega'^2(0). \tag{2.5}$$

Differentiating

$$\frac{\partial F}{\partial y} = f'(g(x) + g(y))g'(y)).$$

and taking into account (2.4) we have

$$\frac{\partial^2 F}{\partial y^2}(x,0) = f''(g(x)) + f'(g(x))g''(0) = (\omega'(x) - \omega'(0))\,\omega(x).$$

So $W_2(x)$ is correctly defined: the left side of (2) divisible by 2. So is the right side and $\omega(x)$ is invertible by definition.

Applying $\frac{\partial^2}{\partial y^2}$ to (1.2) we have

$$\frac{\partial^2 A}{\partial y^2} = \frac{\partial^2 F(x,y)}{\partial y^2}(x\omega(y) - y\omega(x)) + 2\frac{\partial F(x,y)}{\partial y}(x\omega'(y)) - \omega(x)) + F(x,y)x\omega''(y).$$

It follows that $A(x,y)$ modulo $(xy)^3$ equals to

$$\begin{aligned} \frac{\partial^2 A}{\partial y^2}(x,0) = (\omega'(x)\omega(x) - \omega'(0)\omega(x))x+ \\ 2\omega(x)(x\omega'(0)) - \omega(x))+ \\ x^2\omega''(0)). \end{aligned}$$

Thus $A(x,y)$ modulo $(xy)^2$ equals the sum of the first two terms in Theorem 1.2.

The next step is

$$\frac{\partial^3 F}{\partial y^3} = f'''(g(x)+g(y))g'(y)^3 + 3f''(g(x)+g(y))g'(y)g''(y) + f'(g(x)+g(y))g'''(y).$$

$$\frac{\partial^3 F}{\partial y^3}(x,0) = f'''(g(x)) + 3f''(g(x))g''(0) + f'(g(x))g'''(0). \tag{2.6}$$

and taking (2.3) and (2.5)

$$\frac{\partial^3 F}{\partial y^3}(x,0) = \omega^2(x)\omega(x)'' + \omega(x)\omega'^2(x) + 3\omega'(x)\omega(x)g''(0) + \omega(x)g'''(0). \tag{2.7}$$

$$\begin{aligned}
&\frac{\partial^3 F}{\partial y^3}(x,0) = \\
&\omega(x)[\omega(x)\omega(x)'' + \omega'^2(x) - 3\omega'(x)\omega'(0) - \omega(0)'' + 2\omega'^2(0)] = \\
&\omega(x)[\omega(x)\omega(x)'' + \omega'^2(x) - 3\omega'(x)\omega'(0) - \omega(0)'' + 3\omega'^2(0) - \omega'^2(0)] = \\
&\omega(x)[\omega(x)\omega(x)'' + \omega'^2(x) - 3\omega'(0)(\omega'(x) - \omega'(0)) - \omega(0)'' - \omega'^2(0)] = \\
&\omega(x)[\omega(x)\omega(x)'' + \omega'^2(x) - \omega(0)'' - \omega'^2(0) - 6x\omega'(0)W_2(x)].
\end{aligned}$$

So we can define

$$\widetilde{\omega}(x) := \frac{\omega(x)\omega(x)'' + \omega'^2(x) - \omega(0)'' - \omega'^2(0)}{6x}$$

and get

$$\frac{\partial^3 F(x,y)}{\partial y^3}(x,0) = 6x\omega(x)\big(-\omega'(0)W_2(x) + \widetilde{\omega}(x)\big). \tag{2.8}$$

To compute $A(x,y)$ modulo $(xy)^4$ it remains following term

$$\sum_{i>3} A_{ij}x^i y^3 = \frac{1}{6}\frac{\partial^3 A}{\partial y^3}(x,0)y^3 + A_{23}x^2y^3. \tag{2.9}$$

One has

$$\begin{aligned}
\frac{\partial^3 A}{\partial y^3} =& \frac{\partial^3 F(x,y)}{\partial y^3}(x\omega(y) - y\omega(x)) + \\
& 3\frac{\partial^2 F(x,y)}{\partial y^2}(x\omega'(y) - \omega(x)) + \\
& 3\frac{\partial F(x,y)}{\partial y}x\omega''(y) + F(x,y)x\omega'''(y).
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial^3 A}{\partial y^3}(x,0) =& 6x^2\omega(x)\left(-\omega'(0)W_2(x) + \widetilde{\omega}(x)\right) + \\
& 6xW_2(x)\omega(x)(x\omega'(0) - \omega(x)) + \\
& 3x\omega(x)\omega''(0) + \\
& x^2\omega'''(0) = \\
& 6x^2\omega(x)\widetilde{\omega}(x) - 6xW_2(x)\omega^2(x) + 3x\omega(x)\omega''(0) + x^2\omega'''(0).
\end{aligned}$$

Therefore we get by (2.9)

$$\sum_{i\geq 4} A_{ij}x^iy^3 = x^2\omega(x)\widetilde{\omega}(x)y^3 - xW_2(x)\omega^2(x)y^3 + \frac{1}{2}\omega''(0)x\omega(x)y^3$$

Here $A_{23} = w_3/2$, $\omega'''(0) = 6w_3$, $\omega''(0) = 2w_2$.

This implies

$$\sum_{i\geq 4} A_{ij}x^iy^3 = \omega(x)W_3(x)x^3y^3,$$

where

$$x^2W_3(x) = x\widetilde{\omega}(x) - W_2(x)\omega(x) + \frac{1}{2}\omega''(0) + \frac{1}{12}x\omega'''(0).$$

□

3. Proof of Theorem 1.1

Let $W_i(x) \in \Lambda(x)$, $i \geq 2$ be defined by

$$\omega(x)W_i(x)x^ny^n = \sum_{i\geq n} A_{in}x^iy^n = \frac{1}{n!}\frac{\partial^n A}{\partial y^n}(x,0)y^n + A_{n-1n}x^{n-1}y^n + \cdots + A_{2n}x^2y^n. \tag{3.1}$$

We note that $W_i(x)$ thus defined fits into Theorem 1.1: factor $\omega(x)$ is motivated by the initial three terms, which corresponds the Abel formal group law in the introduction. But this factor is invertible as $\omega(0) = 1$, so this is not a problem. So the first equality is a correct definition of $W_i(x)$. The second equality is just the method of undetermined coefficients.

Now we discuss how to work out explicit calculations. We need to compute $\frac{\partial^n A}{\partial y^n}(x,0)$ in terms of $\omega(x)$.

Recall

$$g'(x) = 1 + \mathbf{CP}_1x + \cdots + \mathbf{CP}_kx^k + \cdots,$$

where $\mathbf{CP}_k$ the class represented by the complex projective space of dimension $2k$. The coefficients of $g^{(i)}(x)$ for $i \geq 1$ are in the Lazard ring and

$$g^{(n)}(0) = (n-1)!\mathbf{CP}_{n-1}. \tag{3.2}$$

As we work in the Lazard ring, it is better to compute $f^{(n)}(g(x))$ in terms of $\omega, \omega', \cdots, \omega^{(i)}$ by applying

$$f'(g) = 1/g'(x) = \omega(x);\;\; f^{(n)}(g(x)) = \omega(x)(f^{n-1}(g(x)))'. \tag{3.3}$$

Proposition 3.1. *Let $A(x,y)$, $F(y)$ be as above and $P_{n,k}$ be the Bell polynomials. Then*

$$\frac{\partial^n A}{\partial y^n}(x,0) = -n\omega(x)\frac{\partial^{n-1}F}{\partial y^{n-1}}(x,0) + x\sum_{i=0}^{n} w_i\frac{n!}{(n-i)!}\frac{\partial^{n-i}F}{\partial y^{n-i}}(x,0); \tag{3.4}$$

where

$$\frac{\partial^m F(x,y)}{\partial y^m}(x,0) = \sum_{k=1}^{m} f^{(k)}(g(t))P_{m,k}(\mathbf{CP}_1, \cdots, (i-1)!\mathbf{CP}_{i-1}, \cdots, (m-k)!\mathbf{CP}_{m-k}). \tag{3.5}$$

Proof. We start with (1.1).

For computation of $\frac{\partial^n F(x,y)}{\partial y^n}(x,0)$ we need the chain rule expressed $\frac{\partial^n}{\partial t^n}h(g(t))$ in terms of Bell's polynomials $P_{n,k}$

$$\frac{\partial^n}{\partial t^n}h(g(t)) = \sum_{k=1}^{n} h^{(k)}(g(t))P_{n,k}(g'(t), g''(t), \cdots, g^{(n-k+1)}(t)).$$

So that for $h = f(g(x)+g(y))$ we have

$$\frac{\partial^n F}{\partial y^n}(x,0) = \sum_{k=1}^{n} f^{(k)}(g(x))P_{n,k}(g'(0), g''(0), \cdots, g^{(n-k+1)}(0))$$

and apply (3.2) to get (3.5).

For computation of $\frac{\partial^n A}{\partial y^n}(x,0)$ we apply the Cartan formula for

$$F = F(x,y) \quad \text{and} \quad G = x\omega(y) - y\omega(x)$$

as in (1.2).

$$\frac{\partial^n}{\partial y^n}(F\cdot G)(x,0) = \sum_{0}^{n} \binom{n}{k} \frac{\partial^{n-k} F}{\partial y^{n-k}}(x,0)\frac{\partial^k G}{\partial y^k}(x,0).$$

Finally to get (3.4) note that

$$\begin{aligned} &G(x,0) = x,\\ &\frac{\partial G}{\partial y}(x,0) = xw_1 - \omega(x),\\ &\frac{\partial^k G}{\partial y^k}(x,0) = x\omega_{k-1}(0) = xk!w_k, \text{ for } k\geq 2. \end{aligned}$$

□

4. Proof of Theorem 1.3

Let $A(x,y)$ be the series in (1.2) and g, f and $\omega = 1/g'$ as in Section 2. Note that one has modulo decomposable coefficients

$$F(x,y) = f(g(x)+g(y)) = g(x) + g(y) + \sum f_k(x+y)^{k+1},$$

and as $g(x) = x + \sum \frac{\mathbf{CP}_i}{i+1}x^{i+1}$

$$A(x,y) = (x-y)\Big(\sum \frac{\mathbf{CP}_i}{i+1}x^{i+1} + \sum \frac{\mathbf{CP}_i}{i+1}y^{i+1} + \sum f_{i+j-1}\binom{i+j}{i}x^i y^j\Big) + (x+y)(xw(y)-yw(x)).$$

So that killing $A_{n+1\,j}$, $j > n+1$, i.e., killing the coefficients of W_{n+1}, is equivalent to killing the elements $\{f_k + \text{some decomposable terms}, k > 2n\}$ in $\Lambda\otimes\mathbb{Q}$. But modulo decomposables f_k coincide with $\mathbf{CP}_k$ up to some factor.

As explained above W_{n+1} can be explicitly written in terms of the invariant differential form $\omega_{F_n} = 1/(1+\sum \rho_n(\mathbf{CP}_i)x^i)$. This is convenient for actually constructing a regular sequence and, thus, the cohomology theory that implements F_n.

□

References

[1] M. Bakuradze, *On the Buchstaber formal group law and some related genera*, Proc. Steklov Math. Inst., 286(2014), 7-21.
[2] M. Bakuradze, *Formal group laws by Buchstaber, Krichever and Nadiradze coincide*, Uspekhi Mat. Nauk, **68:3** (2013), 189-190.
[3] M. Bakuradze, *Computing the Krichever genus*, J. Homotopy Relat. Struct, **9, 1**(2014), 85-93.
[4] M. Bakuradze, V. Vershinin, *On addition theorems related to elliptic integrals*, to appear.
[5] V. M. Buchstaber, *Functional equations associated with addition theorems for elliptic functions and two-valued groups,* Uspekhi Mat. Nauk, **3, 273** (1990), 185-186.
[6] V.M. Buchshtaber, A. N. Kholodov, *Formal groups, functional equations and generalized cohomology theories*, (Russian) Mat. Sb. **181, 1** (1990), 75-94.
[7] V. M. Buchstaber, T. E. Panov, *Toric Topology*, Mathematical Surveys and Monographs, **204**, Amer. Math. Soc., 2015.
[8] M. Hazewinkel, *Formal Groups and Applications*, Elsevier, 1978.

[9] M, Lazard, *Sur les groupes de Lie formels à un paramètre*, Bulletin Soc. Math. France,**83**(1955), 251 - 274
[10] S. Ochanine, *Sur les genres multiplicatifs definis par des integrales elliptiques*, Topology **26:2** (1987), 143-151.
[11] R. Nadiradze, "Formal group and cohomology theories", Dissertation for the Doctoral Science Degree, Tbilisi 1995
[12] S. P. Novikov, *The methods of algebraic topology from the viewpoint of cobordism theory*, Math. USSR-Izv. **1:4** (1967), 827-913
[13] D. Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293-1298.
[14] N.-A. Baas *On Bordism Theory of Manifolds with Singularities*, Mathematica Scandinavica, Vol. 33 (1973), 279-302.
[15] V. V. Vershinin, *Cobordism s and spectral sequences*, AMS, Translations of Math. monographs, 130. 1993.

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